# Group Theory (Continue)

Wednesday, October 1, 2025 9:30 AM

Missing
100# 5, 7, 13, 14,
Here 208
200# 1, ~~8~~, 17, 24,
Ex #227 on 9/22
500 # 2, 14, 18, 19, 21, 23, 25, 26, 29
Ex #511 on 9/15, 17

## Recall: Groups

6] Def$^n$.

The group is abelian (or commutative) if

$$\forall a, b \in G, \quad ab = ba$$

7] e.g. ① $(\mathbb{Z}_{10}, +)$ and $(\mathbb{Z}_5^*, \cdot)$ are abelian groups

e.g. ② Is $(A, *)$ abelian

if $A = \{1, -1, i, -i, j, -j\}, k, -k\}$
and $*$ is defined by:

| $*$ | $1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ |
|---|---|---|---|---|---|---|
| $1$ | $1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ |
| $-1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ | $1$ |
| $i$ | $i$ | $-i$ | $j$ | $-j$ | $1$ | $-1$ |
| $-i$ | $-i$ | $j$ | $-j$ | $1$ | $-1$ | $i$ |
| $j$ | $j$ | $-j$ | $1$ | $-1$ | $i$ | $-i$ |
| $-j$ | $-j$ | $1$ | $-1$ | $i$ | $-i$ | $j$ |

Yes abelian

8] Def$^n$ (Group order)

the group $G$ is finite if it has a finite number of elements ; called the order of $G$, denoted $|G|$

9] Def$^n$. (The order of a)

let $a \in G$ if there is a positive number $n$ s.t. $a^n = e$, then $a$ has a finite order, and the smallest such $n$ is called the order of $a$, denoted by ord(a).

If there is no such $n$, then $a$ has an infinite order.

e.g. in $\mathbb{Z}_{10}$, $\text{ord}(2) = 5$ for $2+2+2+2+2=0$

$\text{ord}(3) = 10$ for $10(3) = 0$

$\text{ord}(6) = 5$ for $5(6) = 0$

in $(\mathbb{Z}, +)$, 2 has an infinite order..

in $(\mathbb{Z}_{11}^*, \cdot)$ $\text{ord}(10) = 2$ for $10^2 = 1$

$\text{ord}(2) = 10$ for $2^{10} = 1$

$\text{ord}(9) = 5$ for $9^5 = 1$

$\text{ord}(4) = 5$ for $(2^2)^5 \equiv 1 \pmod{11}$

$\text{ord}(3) = 5$

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Review HW 2

Q9. (6) $3456789 \cdot 34567 \pmod 9$

$\equiv 6 \cdot 7$

$\equiv 6 \pmod 9$

(c) $2640 \cdot 3928 \pmod{13}$

$\equiv 1 \cdot 2 \equiv 2 \pmod{13}$

Q8.

for $k=2$ $\quad n=2$

$k=3$ $\quad =4$

$k=11$ $\quad n=2^{10}=1024$

$k=15$ $\quad n=2^{14}=4k$ or $n=2^4 \cdot 3^2 = 144$

for $n=144$

| | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ |
|---|---|---|---|---|---|
| $3^0$ | 1 | 2 | 4 | 8 | 16 |
| $3^1$ | 3 | 6 | 12 | $2^i 3^j$ | — |
| $3^2$ | 9 | — | — | — | 144 |

$2^i \cdot 3^j \mid 144$ for $i = 0 \dots 4$, $j = 0 \dots 2$

$2^1 \cdot 3^1 \cdot 5^1 = 30$

Q8. smallest $n$ with $k$ div.

For $k=1$ $\quad n=1$

$k=2$ $\quad n=2$

$k=3$ $\quad n=4$

$k=11$ $\quad n=2^{10} = 1024$

$\Longrightarrow 2^i \mid 1024$ for $i=0\ldots10$

$k=15 = 5\cdot 3$ $\quad n=2^{14} = \underline{\underline{16\times1024}}$ or $n=2^4\cdot 3^2$ $= 144$ smaller

$2^i\cdot 3^j \mid 144$ for $i=0\ldots4$, $j=0\ldots2$

| | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ |
|---|---|---|---|---|---|
| $3^0$ | 1 | 2 | 4 | 8 | 16 |
| $3^1$ | 3 | 6 | 12 | — | — |
| $3^2$ | 9 | — | 36 | $2^i\cdot3^j$ | 144 |

$k=10$ $\quad n=2^9 = 512$

or $n=2^4\cdot 3^1 = 48$

$\therefore$ 48 has exactly 10 divisor

$\underline{k = 8}$

$n = 2^7 = 128$

8 $\longrightarrow$ $n = 2^3 \cdot 3^1 = 24$ smallest

$4 \times 2$

$2 \times 2 \times 2 \longrightarrow n = 2^1 \cdot 3^1 \cdot 5^1 = 30$

$2 \times 4 \longrightarrow n = \underline{5}^1 \cdot \underline{7}^3$ also has 8 divisor

$n = 10^7$ ✗ has 64 divisors

## Q6

$$3x + 2y \equiv 5 \pmod 7 \quad \text{--- (1)}$$
$$4x + 6y \equiv 4 \pmod 7 \quad \text{--- (2)}$$

$+$

$$0x + y \equiv 2 \pmod 7 \Rightarrow y \equiv 2$$

by sub. in (2)

$$4x + 6(2) \equiv 4$$
$$4x - 2 \equiv 4$$
$$x \equiv 6 \pmod 7$$

$$\therefore x \equiv 4^{-1} \cdot 6 \equiv 2 \cdot 6 \equiv 5 \pmod 7$$

## Q9.

$$\cancel{26}40 \times \cancel{39}28 \pmod{13}$$

$$\equiv 1 \cdot 2 \equiv 2 \pmod{13}$$

## Q7.

$$7x \equiv 4 \pmod 9$$

$$\Rightarrow \quad x \equiv 7^{-1} \cdot 4$$
$$\equiv 4 \cdot 4$$
$$\equiv -2 \equiv 7 \pmod 9$$

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HW 2 Review (540)

1 8. smallest $n$ has $k$ divisors?

for $k=1$, $n=1$

$k=2$, $n=2$

$k=3$, $n=4$

$k=11$, $n = 2^{10} = 1024 = 1K$

$2^i \mid 1024$ for $i = 0 \cdots 10$

$k = 15 = 5 \times 3$

$n = 2^{14} = 2^4 \cdot 2^{10} = 16K$

$n = 2^4 \cdot 3^2 = 144$ smaller

$2^i \cdot 3^j \mid 144$ for $i = 0 \cdots 4$, $j = 0 \cdots 2$

| | $2^0$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ |
|---|---|---|---|---|---|
| $3^0$ | 1 | 2 | 4 | 8 | 16 |
| $3^1$ | 3 | 6 | 12 | 24 | 48 |
| $3^2$ | 9 | 18 | 36 | $2^i \cdot 3^j$ | 144 |

For $k = 10 = 5 \times 2$

$n = 2^9 = 512$

or $n = 2^4 \cdot 3^1 = 48$ smaller

For $k = 8$

$= 2 \times 2 \times 2$

$= \underline{\underline{4 \times 2}}$ ✓

$= 2 \times 4$ ✗

$n = 2^7 = 128$

or $n = 2^1 \cdot 3^1 \cdot 5^1 = 30$

or $n = 2^3 \cdot 3^1 = 24$ smallest

$n = 2^1 \cdot 3^3 = 54$

$10^2 = 2^2 \cdot 5^2 = 100$

$2^{0,1,2} \cdot 5^{0,1,2}$

$1000 = 2^3 \cdot 5^3$

$\underline{\underline{18^2}} = 2^2 \cdot 3^4$

$|\mathbb{Z}_n^*|$