Grover Algorithm

Saturday, November 19, 2022 2:22 PM

§ 6.3. Grover's Algorithm

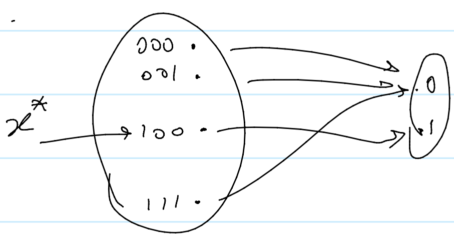
1] Search Problem:

Given $f: \{0,1\}^n \longrightarrow \{0,1\}$, $\exists \hat{x}$, $f(\hat{x}) = 1$

it has exactly one input x^* s.t. $f(x^*) = 1$

$$f(\hat{x}) = \begin{cases} 1 & \text{if } \hat{x} = x^* \\ 0 & \text{if } \hat{x} \neq x^* \end{cases}$$

e.g.



2] Classical Approach

Check all inputs: for $i = 0$ to $N = 2^n$

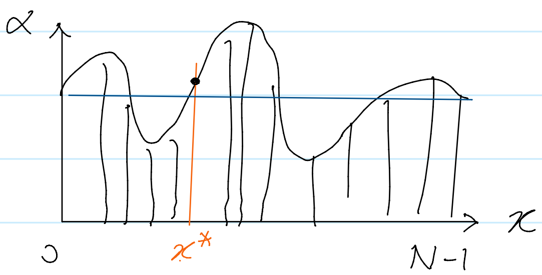
$\Longrightarrow$ Time complexity $= O(2^n) \Rightarrow$ Exponential

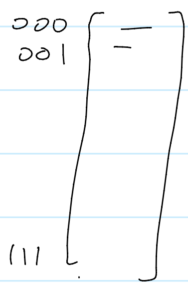
3] Quantum Approach (by Grover)

- We can find x^* in $O(\sqrt{N}) = O(2^{n/2})$ evaluations of f.
- Grover's algorithm has 2 steps
  1. Phase Inversion (PI)
  2. Inversion about the mean (MI)

4] Phase Inversion

$f(x^*) = 1$





$\psi = \sum_x \alpha_x |x\rangle$

$2^n = N$

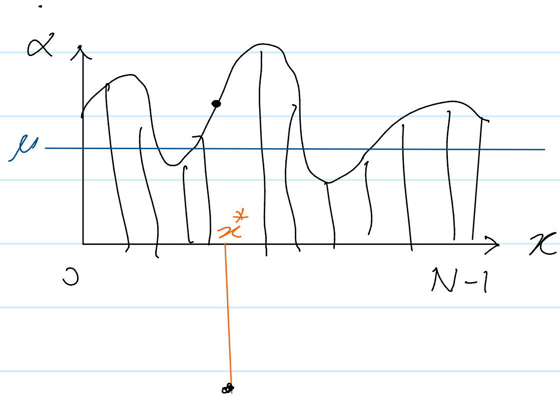
for ψ_n, $\alpha_x = \frac{1}{\sqrt{\ }}$

for ψ_0 , $\alpha_x = \frac{1}{\sqrt{N}}$ $\qquad\qquad$ $2^n = N$

Phase Inversion:

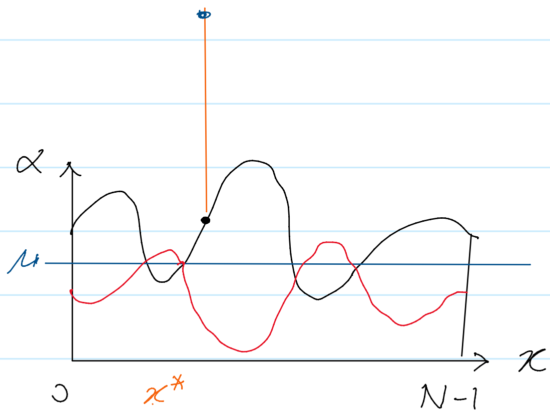
$$\psi \xmapsto{PI} \sum_{x \neq x^*} \alpha_x |x\rangle - \alpha_{x^*} |x^*\rangle$$

$$(-1)^{f(x)}$$



$$f(x) \rightarrow \begin{cases} +1|x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{N}}\right], & \text{if } x \neq x^* \\ -1|x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{N}}\right], & \text{if } x = x^* \end{cases}$$

5] Inversion about the mean



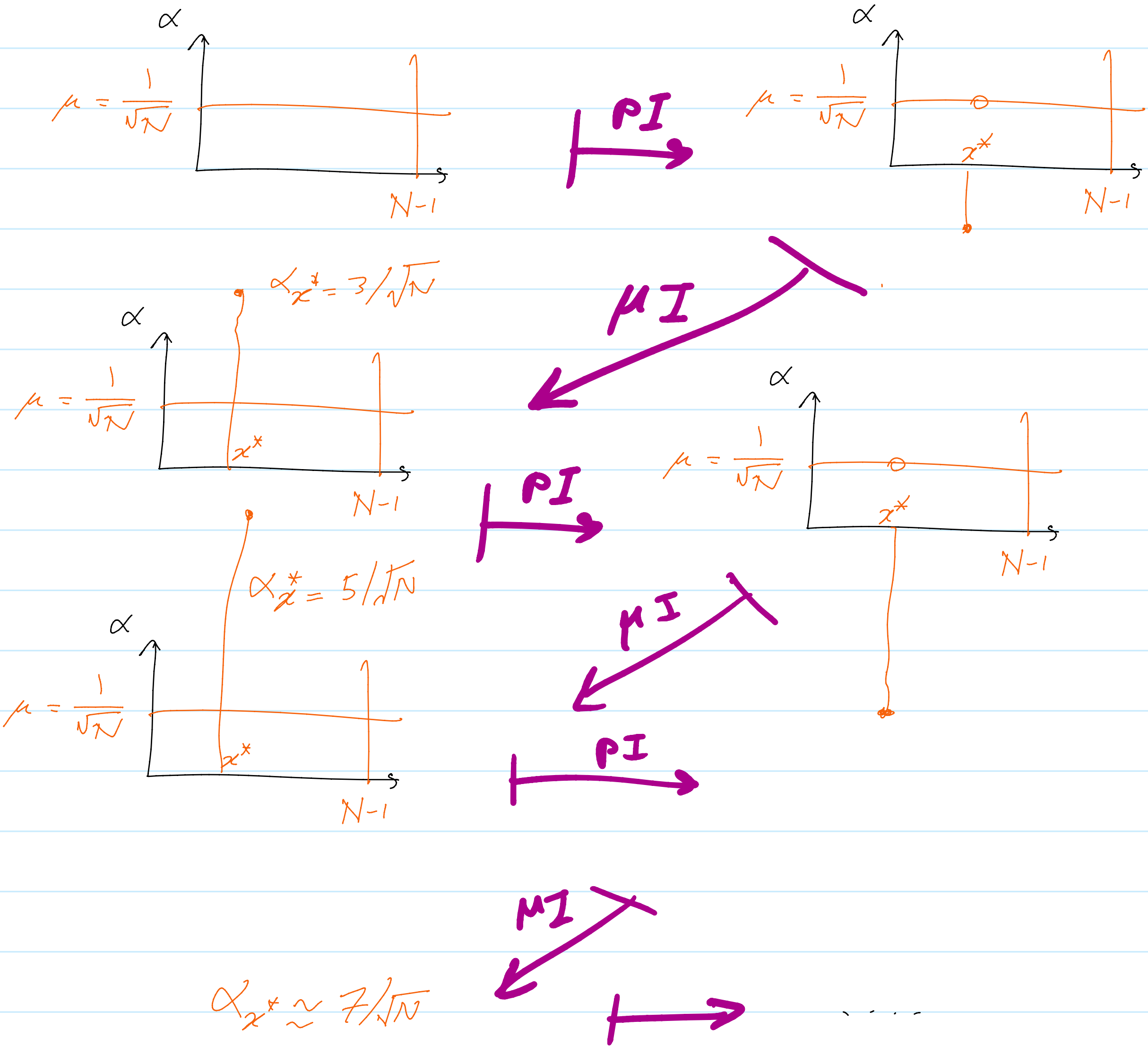
let μ be the mean

$$\mu = \frac{\sum_{x=0}^{N-1} \alpha_x}{\quad}$$

$$\alpha_x \xmapsto{MI} \mu + (\mu - \alpha_x) = 2\mu - \alpha_x$$

$$\sum_x \alpha_x |x\rangle \xmapsto{MI} \sum_x 2\mu - \alpha_x |x\rangle$$

6] Grovers Algorith (Idea)



$$\frac{\sqrt{N/2}}{2} \text{ steps}$$

$$\frac{\sqrt{N/2}}{\sqrt{N}} = \frac{1}{\sqrt{2}}$$

See the slides on Grover Algorithms