MatrixChain Algorithm

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Recall : Matrix - Chain Problem

$$M_1 \cdot M_2 = M_3$$

$$(n \times m) \cdot (m \times k) \quad \underbrace{(n \times k)}_{n \cdot m \cdot k \text{ mult ops.}}$$

1] Matrix - Chain Problem :

$$M_1 \cdot M_2 \cdot \; \dots \; \cdot M_n$$

dim: $(r_1 \times r_2) \cdot (r_2 \times r_3) \cdot \; \dots \; \cdot (r_n \times r_{n+1})$

Problem Statement : Given $n+1$ dimensions $r_1, r_2, \dots, r_{n+1}$
Find the minimum cost of multiplying the n matoices of the given dimentions.

2] Sub problem :

Consider a sub chain of the main chain:

$$M_{i,j} = M_i \cdot M_{i+1} \cdot \; \dots \; \cdot M_j$$

Sub-solution :

let $C[i,j]$ be the cost of multiplying $M_{i,j}$

$$\boxed{\underset{r_1 \times r_2}{\boxed{M_1}} \cdot \underset{r_2 \times r_3}{\boxed{M_2}}} \cdot \boxed{\underset{r_3 \times r_4}{\boxed{M_3}} \cdot \underset{r_4 \times r_5}{\boxed{M_4}}} = M_{1,4}$$

$$(r_1 \times r_3) \qquad (r_3 \times r_5)$$

$$C[1,2] + C[3,4] + r_1 \cdot r_3 \cdot r_5$$

$$\Rightarrow C[i,j] = \min_{i<k\le j} \{ c[i,k-1] + c[k,j] + \underbrace{r_i \cdot r_k \cdot r_{j+1}}_{\text{cost of last mult.}} \}$$

3] To compute the cost of $M_{1,n}$, we need $c[1,n]$ mult ops.
this has many overlapping recursive calls.

4] Dynamic - Programming: Matrix Chain Algorithm

$$C[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i<k\le j} \{ c[i,k-1] + c[k,j] + r_i r_k r_{j+1} \} & \text{if } i<j \end{cases}$$

e.g. input : 5, 10, 4, 6, 10, 2

M_1	M_2	M_3	M_4	M_5
(5x10)	(10x4)	(4x6)	(6x10)	(10x2)

Fill the 2D array

	1	2	3	4	5
1	0	200	320	620	348
2		0	240	640	248
3			0	240	168
4				0	120
5					0

$C[1,1] = C[2,2] = \cdots = C[5,5] = 0$

$c[1,2] = 5 \cdot 10 \cdot 4 = 200$

$\vdots$

$c[4,5] = 6 \cdot 10 \cdot 2 = 120$

$$C[1,3] = \min_{k=2,3} \{ 0 + 240 + 5 \cdot 6 \cdot 10 = 540, \; 200 + 0 + 5 \cdot 4 \cdot 6 = 320 \} = 320$$

$$c[2,4] = \left.\begin{array}{l} 240 + 10\cdot 4\cdot 10 = 640 \\ 240 + 10\cdot 6\cdot 10 = 840 \end{array}\right\} \Rightarrow 840$$

$$c[3,5] = 120 + 48 = 168$$

$$\min_{i<k\le j} \{ c[i,k-1] + c[k,j] + r_i r_k r_{j+1} \} \quad \text{if } i<j$$

$$\begin{aligned} c[1,4] &= 0 + 640 + 5\cdot 10\cdot 10 = 1140 \\ &= 200 + 240 + 5\cdot 4\cdot 10 = 640 \\ &= 320 + 0 + 5\cdot 6\cdot 10 = 620 \xrightarrow{\min} 620 \end{aligned}$$

$c[2,5] = \ldots\ldots = 248$

$c[1,5] = \ldots = 348$

Return $c[1,5]$

$\therefore$ the min cost is 348 multi ops.

6] Time Complexity :

$$\sum_{d=1}^{n-1} \sum_{i=1}^{n-d} \sum_{k=1}^{d} c = \frac{cn^3 - cn}{6}$$

$$= \Theta(n^3) \text{ mult. ops.}$$

7] Space Complexity :

$$\Theta(n^2)$$