Master Theorem

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Solving Recurrence Relations
- by expansion
- by the master theorem

1) Expansion

e.g.

- Time complexity analysis of MergeSort:

  $T(n) = 2\,T(n/2) + \Theta(n) = 2\,T(\frac{n}{2}) + c\cdot n$

- By solving this recurrence relation, we find:

  $T(n) = \Theta(n \log n)$

By expansion

$n = 2^k \Rightarrow k = \log n$

$$
\begin{aligned}
T(n) &= 2\,T\left(\tfrac{n}{2}\right) + c\cdot n \\
&= 2\left[2\,T\left(\tfrac{n}{4}\right) + c\cdot\tfrac{n}{2}\right] + c\cdot n = 2^2\left[T\left(\tfrac{n}{2^2}\right)\right] + 2n \\
&= 2^2\left[2\,T\left(\tfrac{n}{2^3}\right) + \tfrac{n}{2^2}\right] + 2n = 2^3\,T\left(\tfrac{n}{2^3}\right) + 3n \\
&\;\vdots \\
&= 2^i\,T\left(\tfrac{n}{2^i}\right) + i\cdot n \\
&= 2^k\cdot T(1) + k\cdot n \\
&= n\cdot 1 + (\log n)\cdot n = \theta(n \log n)
\end{aligned}
$$

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$\log_2 8 = 3$

2) The master theorem

- **The Master Theorem:**

Let $a \geq 1$ and $b > 1$ be two constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence:

$$T(n) = a\,T(n/b) + f(n)$$

then $T(n)$ can be bounded asymptotically as follows

If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.

If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $a f(n/b) \leq c f(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

let $R = \log_b a$

$$n^{R} \quad \text{vs} \quad f(n)$$

Case 1

$f(n) < n^{R}$

$f(n) = O(n^{R-\varepsilon})$

$T(n) = \theta(n^{R})$

2

$f(n) = \theta(n^{R})$

$T(n) = (n^{R} \log n)$

e.g. merge sort

$$T(n) = 2\,T\left(\frac{n}{2}\right) + \underbrace{c \cdot n}_{\theta(n)}$$

$$n^{\log_b a} = n^{1}$$

$$f(n) = c \cdot n = \theta(n^{1})$$

$$\Rightarrow T(n) = \theta(n^{1} \log n)$$

3

$f(n) > n^{R}$

$f(n) = \Omega(n^{R+\varepsilon})$

$T(n) = \theta(f(n))$