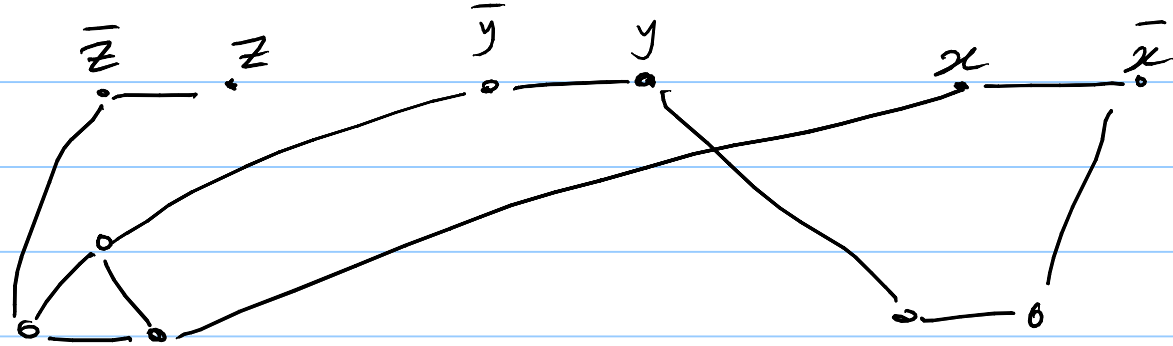
NP-Complete Problems

Note Title 9/15/2019

Recall: SAT α_p VC

Construct:

G:



$k = 3 + 2 + 1 = 6$

$$k = n + \sum_{j=1}^{m} (|C_j| - 1)$$; where C_j is the j^{th} clause.

$\therefore$ f is satisfiable iff G has VC of size k

1] Reduction : SAT α_p VC

Given an instance I of SAT : $f = \underline{\underline{C_1}} \wedge C_2 \wedge \cdots \wedge C_m$ with m clauses and n variables.

1. Construct $G = (V, E)$; $V = V_1 \cup V_2$

$V_1 = \{x_1, x_2, \dots\}$ of all $2n$ literals

$$V_2 = \bigcup_{i=1}^{m} \{v_1^i, v_2^i, \dots, v_{|C_i|}^i\} \Rightarrow |V| = 2n + \sum_{j=1}^{m} |C_j|$$

$$E = \{(x_i, \bar{x}_i) \mid i = 1 \dots n\} \cup \bigcup_{i=1}^{m} \{(v_j^i, v_k^i) \mid j \neq k\}$$

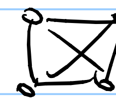
$$\cup \bigcup_{i=1}^{m} \{(x_j, v_k) \mid \text{for every vertix } v \in C_i$$

$$\Rightarrow |E| = n + \sum_{i=1}^{m} \left(\frac{|C_i|(|C_i| - 1)}{2} \right) + \sum_{i=1}^{m} |C_i|$$

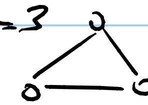
2. Let $K = n + \sum_{i=1}^{m} (|C_i| - 1)$

Complete graph K_n

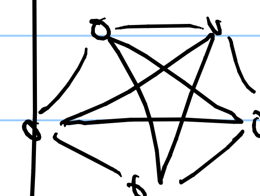
$n = 4$



$n = 3$



$n = 5$



$$|E| = \frac{n(n-1)}{2}$$

2] Thrm: $VC \in NP$-complete

proof: 1. $VC \in NP$

1. a solution can be guessed in $O(n)$ time by choosing a subset S of V, with $|S| = k$
2. Then S can be verified in $O(n^2)$ as follows:
   - for all $(u,v) \in E$, check that $u \in S$ or $v \in S$

2. Since $SAT \in NP\text{-}C$ and $SAT \propto_p VC$, we have $VC \in NP\text{-}C$. $\square$

3) Clique $\in$ NP-Complete

proof: ① Clique $\in$ NP

1. guess a subset $C \subseteq V$ in $O(n)$, with $|C| = r$

2. verify that $C = K_r$ i.e. is C a complete graph

for $i = 1$ to $(r-1)$

for $j = i+1$ to r

for $(i,j) \in C$ check $(i,j) \in E$

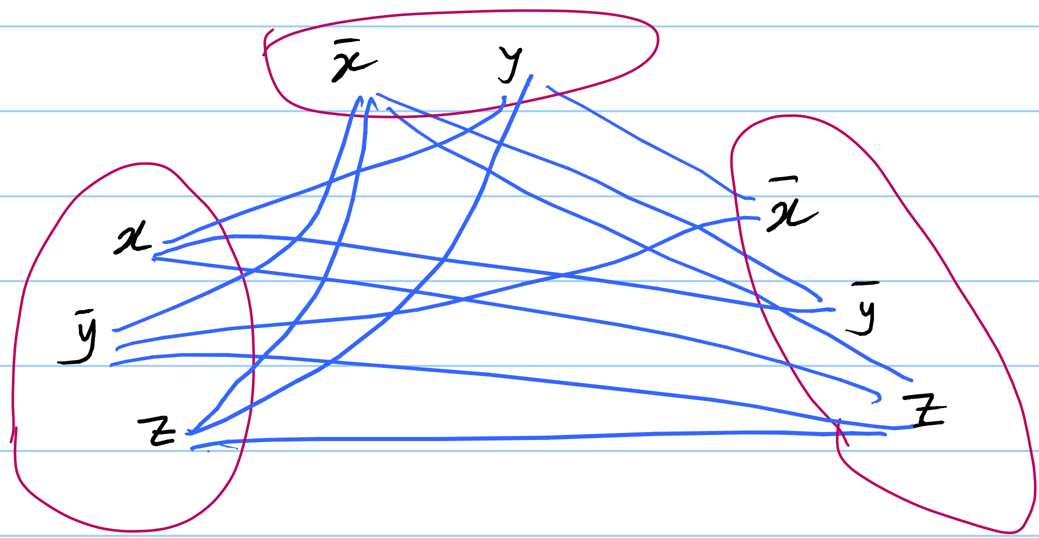
$\Rightarrow O(n^2)$

② show that SAT $\propto_p$ Clique.

See the idea below

4] SAT $\propto_p$ Clique

idea: $f = (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee y) \wedge (\bar{x} \vee \bar{y} \vee z)$



$(u,v) \in E$ iff $u \neq \bar{v}$

u,v from different clauses

SAT(f) iff Clique (G, m)

where m is number of clauses

5] Reduction: SAT $\propto_p$ Clique

Input: $f = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

output: $G = (V, E)$; k

1. $V = \{ x \mid x$ is a literal in $f \}$
2. $E = \{ (u, v) \mid u \in C_i$ and $v \in C_j$, $i \neq j$, $u \neq \bar{v} \}$
3. $k = m$

Then SAT(f) iff Clique(G, k)

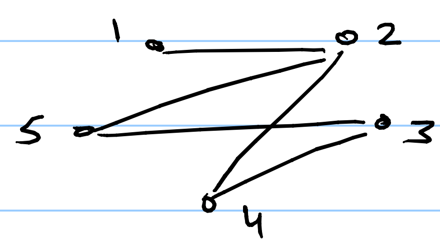
6] Independent Set (IS):

Given $G = (V, E)$, and int $k > 0$, is there

$S \subseteq V$ with $|S| = k$ s.t.

$\forall (u, v) \in S \times S$, $(u, v) \notin E$.

e.g. G:



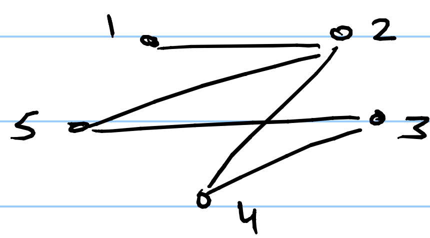
$S_1 = \{1, 3\}$

$S_2 = \{1, 5, 4\}$, $k = 3$

7] $VC \propto_p IS$: the proof is straightforward.

Lemma: let $G = (V, E)$, then $S \subseteq V$ is IS iff $(V - S)$ is a VC in G

e.g. G:

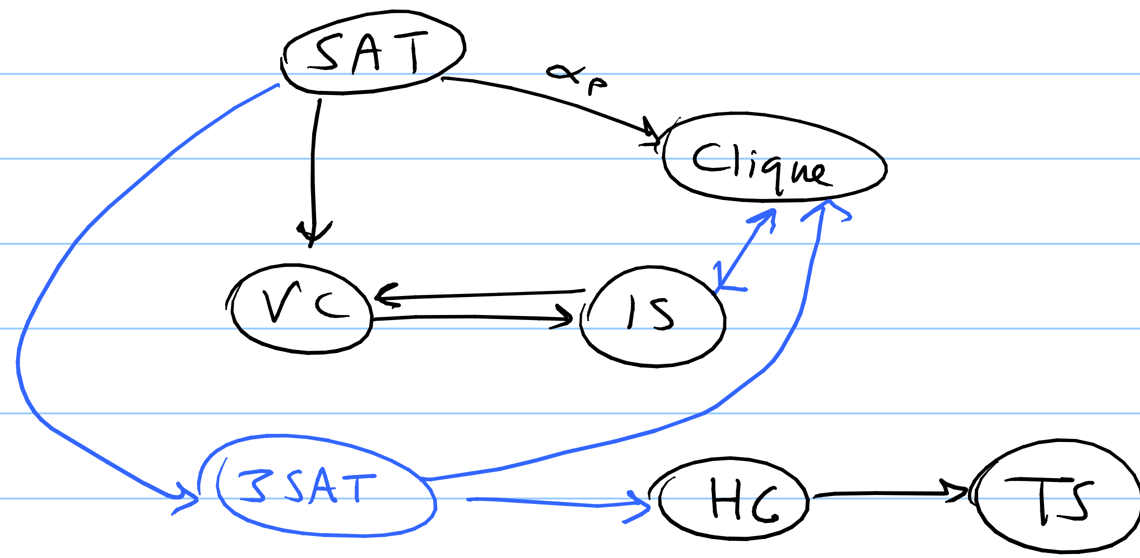


$S_1 = \{1, 3\} \implies \{2, 5, 4\}$ is VC

$S_2 = \{1, 5, 4\}$, $k = 3$

$\implies \{2, 3\}$ is VC

8] NP-Complete Problems



§ 9.5. Class Co-NP

9/17/2019

Recall: Class NP

(The solution "witness" is verifable in polynomial time)

9] Consider these two problems:

1) Unsat: Given a boolean formula f, is f unsatisfiable?

* is Unsat $\in$ NP? for what witness?

2) Tautology: Given a boolean formula, f, is it a tautology?

i.e. if f true for all truth-assignments?

* is TAUT $\in$ NP? for what witness?

10] Def^n. Class Co-NP : a set of all problems whose complement are in NP.

e.g. Unsat $\in$ Co-NP

11] Complement of TS ($\overline{TS}$) : Given G and k, is it true that there is no tour of length k or less ($\leq k$)?

12] Note: $\overline{TS} \in$ Co-NP, since TS $\in$ NP

13] Class Co-NP-Complete (Co-NPC)

a problem Π is Co-NPC if

1) $\Pi \in$ Co-NP, and

2) $\forall\ \Pi' \in$ Co-NP, $\Pi' \propto_p \Pi$

14] Thrm: $\Pi \in$ NPC iff its complement $\overline{\Pi} \in$ Co-NPC

e.g. SAT $\in$ NPC

$\therefore$ Unsat $\in$ Co-NPC

15] Thrm: f is unsatisfiable <u>iff</u> $\overline{f}$ is a tautology

16] TAUT $\in$ Co-NPC, proof from [7] and [8]

17] Note: TAUT $\in$ P iff Co-NP = P

18] Note: TAUT $\in$ NP iff Co-NP = NP

19] Thm: if Π and $\overline{\Pi} \in$ NPC, then Co-NP = NP

20] Note: There are problems in NP and their complements are also in NP: (a.k.a NPI)

e.g. Is-prime? and Is-composite?

21] Relationship between Complexity Classes (CLRS § 34)

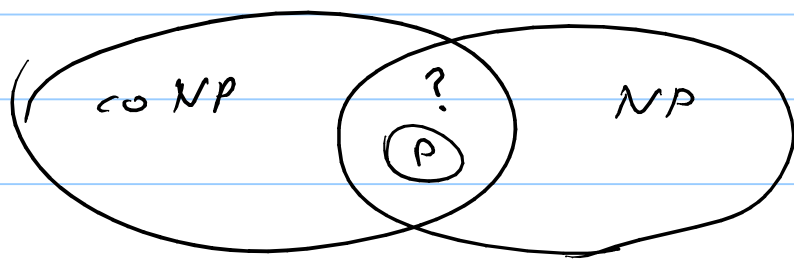
$$NPC \subseteq NP$$

$$\left.\begin{array}{l} P \subseteq NP \\ P \subseteq Co\text{-}NP \end{array}\right\} \; P \subseteq (NP \cap Co\text{-}NP) = NPI$$

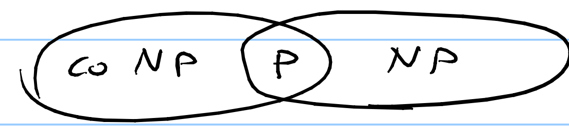
$$CoNPC \subseteq CoNP$$

4 Possibilities:

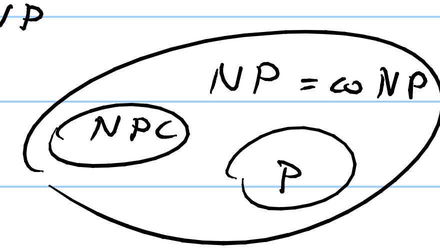
① $P \subset NP \cap coNP$



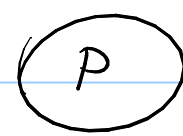
② $P = NP \cap coNP$



③ $NP = coNP$



④ $P = NP = coNP$



22] Circuit-SAT :

Given an instance I of an NP problem Π.

I can be hardwired into a circuit C decides I.

i.e. $C = \text{True}$ iff $I = \text{yes}$

23] Thrm : Circuit-SAT $\propto_p$ SAT

SAT : f

$\therefore \ \forall \Pi \in NP, \ \Pi \propto_p SAT \implies SAT \in NP\text{-hard}$

24] 3-CNF-SAT (3SAT)

3SAT: f is in a Conjunctive Normal Form (CNF)

where each clause has exactly 3 literals

25] SAT $\propto_p$ 3SAT

Reduction: SAT $f = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

① if $C_i = (x_1 \vee x_2) \xrightarrow{\text{replace}} (x_1 \vee x_2 \vee x_1)$

or $(x_1 \vee x_2 \vee z) \wedge (x_1 \vee x_2 \vee \bar{z})$

② if $C_i = (x_1) \longrightarrow (x_1 \vee y \vee z) \wedge (x_1 \vee y \vee \bar{z}) \wedge \cdots$

③ if $C_i = (x_1 \vee x_2 \vee x_3 \vee x_4 \vee \cdots)$

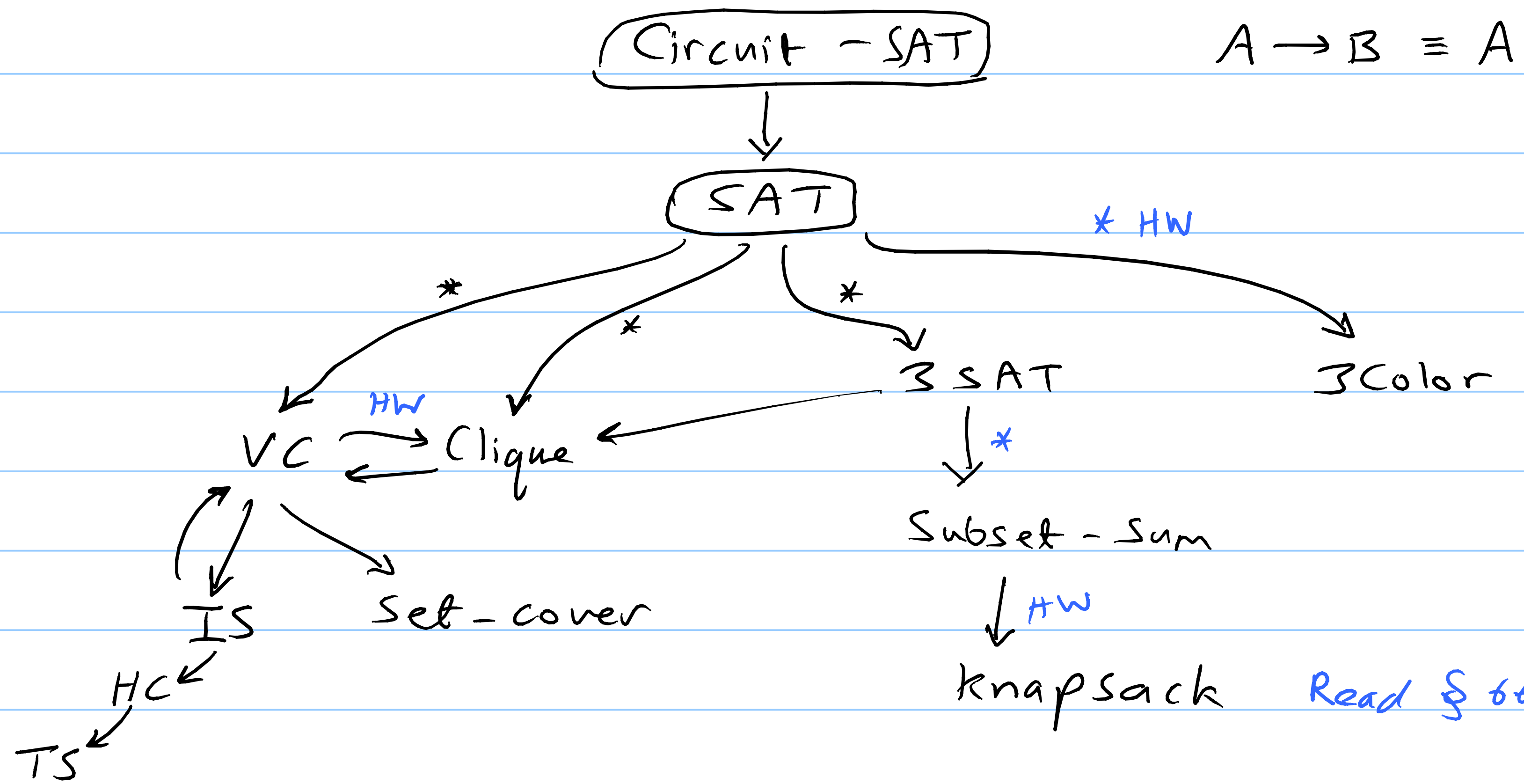
$\longrightarrow (x_1 \vee x_2 \vee z_1) \wedge (\bar{z}_1 \vee x_3 \vee z_2) \wedge (\bar{z}_2 \vee x_4 \vee z_3) \cdots$

$\cdots (\bar{z}_k \vee x_{n-1} \vee x_n)$

$\Rightarrow$ 3SAT $\in$ NPC.

26] Reduction of NP-Complete Problems:

$A \rightarrow B \equiv A \propto_p B$



Read § 66

27] Subset-Sum Problem (SS)

Given a set S of n positive integers, and t,

is there a subset $S' \subseteq S$ s.t.

$$\sum_{x \in S'} x = t$$

e.g.

$S = \{1, 7, 12, 13, 20\}$, $t = 39$

yes $S' = \{7, 12, 20\}$