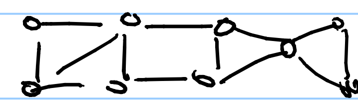
NP-Completeness

Recall: $A \propto_p B$

1] Hamiltonian Cycle (HC) : a cycle that visits each vertex exactly once. e.g.



No HC.

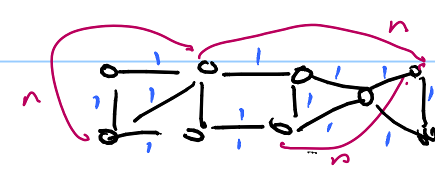
2] Traveling Salesman (TS) : Given n cities, with distances, and k, is there a tour of length $\le k$?

Tour: a cycle visiting each city exactly once.

3] Thrm : $HC \propto_p TS$

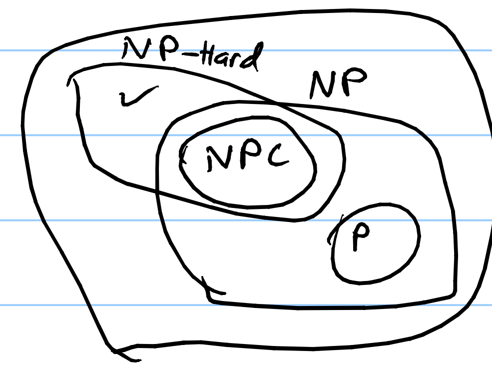
Given HC: $G = (V, E) \longrightarrow G' = (V, E')$, $E' = V \times V$, $w(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E \\ n & \text{otherwise} \end{cases}$

$k = n$;



4] NP-Hard :

Problem A is NP-hard if $\forall \pi \in NP$, $\pi \propto_p A$



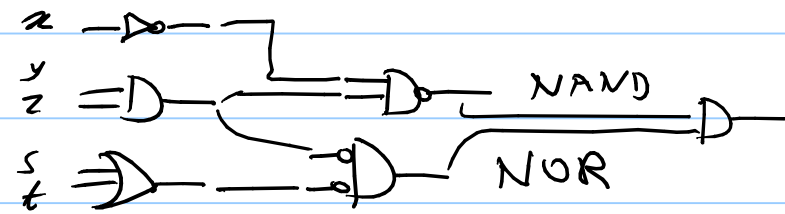
5] NP-Complete :

problem A is NP-Complete if

① $A \in NP$

② A is NP-hard; $\forall \pi \in NP$, $\pi \propto_p A$.

e.g. SAT $\in$ NPC (see proof)

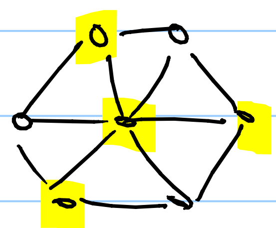


6] SAT: Given a boolean formula f in conjunctive normal form (CNF), is f satisfiable:

e.g. $f = \underbrace{(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_3})}_{3 \text{ clauses}} \in SAT$

7] Vertex Cover (VC) Given undirected graph $G = (V, E)$, and k;
is there $C \subseteq V$ of size k s.t. $\forall (x,y) \in E$, $x \in C$ or $y \in C$.

e.g. G:



VC$(G, 4)$ has a cover of size $k = 4$

8] Independent set (IS) Given G, and k, is there $S \subseteq V$ of size k
s.t. $\forall\ x, y \in S$, $(x, y) \notin E$.

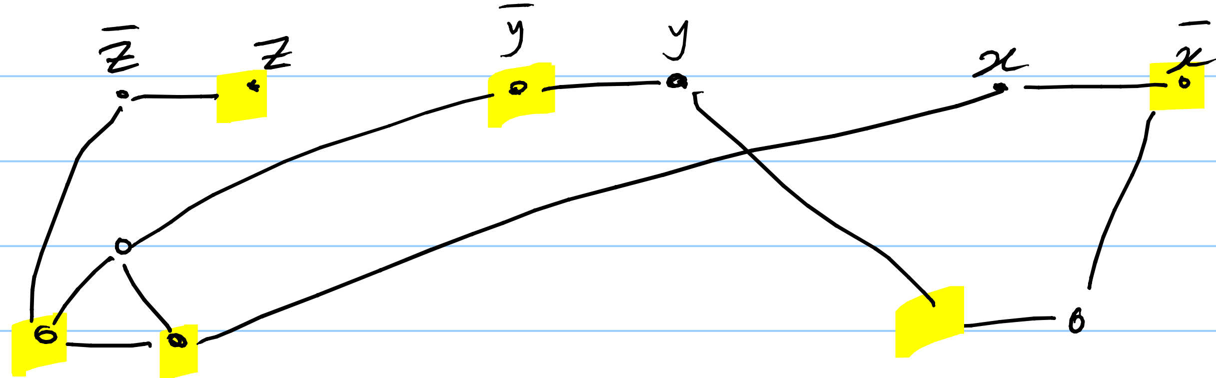
9] Thm: if $A \in NP$, $B \in NPC$, and $B \propto_p A$,
then $A \in NPC$.

10] SAT $\propto_p$ VC

e.g. Given $f = (\overset{F}{x} \vee \overset{T}{\bar{y}} \vee \overset{F}{\bar{z}}) \wedge (\overset{T}{\bar{x}} \vee \overset{F}{y})$

Construct:

G:



$k = 3 + 2 + 1 = 6$

$$k = n + \sum_{j=1}^{m} (|C_j| - 1) \; ; \quad \text{where } C_j \text{ is the } j^{th} \text{ clause.}$$

$\therefore$ f is satisfiable iff G has VC of size k