ICS 553 - Design and Theory of Algorithms

Lecture Notes on Discrete Probability

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§ 1. Introduction

[1] Sample space:
The sample space is the set S of all possible outcomes. |S| is the size of the sample space which means the number of all possible outcomes.

[2] Example:
(a) When a die is rolled, S = {1, 2, 3, 4, 5, 6}. So, there are |S| = 6 possible outcomes.
(b) When 2 dice are rolled, |S| = 36 outcomes.
(c) For 2 coins, |S| = 4 outcomes, namely {HH, HT, TH, TT}
(d) For a hand of five cards from a deck of 52 cards, |S| = C(52, 5)

[3] The probability of an event:
The event E is a subset of S. The probability of E is p(E) = |E| / |S|

[4] Examples:
1. A die is rolled. What is the probability that the outcome is 5 or 6?
2. A die is rolled. What is the probability that the outcome is even?
3. Two dice are rolled. What is the probability that the sum is 7.

Solutions:
1. We have |E| = 2, |S| = 6. Therefore, p(E) = 2/6 = 1/3.
2. Here, |E| = 3, so p(E) = 3/6 = 1/2.
3. E = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}. So we have |E| = 6 and |S| = 36. Therefore, p(E) = 6/36 = 1/6.

[5] Example:
A card is picked at random from a deck of 52 cards. There are 4 suits (spades, clubs, hearts, and diamonds), each has 13 kinds (also called ranks, face values and denominations). What is the probability that it is:
(a) the Ace of spades?
(b) any Queen?
(c) a Queen or a King?
(d) a card of hearts?
(e) a card of clubs or diamonds.

Solution:
(a) |E| = 1, so p(E) = 1/52
(b) |E| = 4, so p(E) = 4/52
(c) |E| = 8, so p(E) = 8/52
(d) |E| = 13, so p(E) = 13/52 = 1/4
(e) |E| = 13 + 13 = 26, so p(E) = 26/52 = 1/2

[6] Sampling:
When two or more cards are drawn from the deck, there are two types of sampling:
1. sampling without replacement – the cards are not returned once they are drawn from the deck (like poker hands)
2. sampling with replacement - the drawn card is returned to the deck before the next card is picked.

[7] Example:
A bag contains 5 balls in different colors. Two balls are picked up randomly. What is the probability that the balls are red and blue if they are picked:
(a) with replacement?
(b) without replacement?

Solution:
(a) $|S| = 5 * 5 = 25$, so p({red, blue}) = 1/25.
(b) $|S| = C(5, 2) = 5 * 4 = 20$, so p({red, blue}) = 1/20.

[8] Combinations of Events:
Theorems: $p(E') = 1 - p(E)$
$p(E \cup F) = p(E) + p(F) - p(E \cap F)$

[9] Example:
What is the probability that a hand of 5 cards from a deck of 52 cards contains 4 cards of same kind?

Solution:
$|S| = C(52, 5)$
$|E| = C(13, 1) * C(4,4) * C(48, 1)$; i.e. choose one kind from 13 kinds, then from the 4 suits of that kind choose all of them, then choose the fifth card from the 48 cards left.
$\therefore p(E) = 13 * 1 * 48 / C(52,5)$

§ 2. Probability Theory

[1] Remarks:
- $p(E) = |E| / |S|$
- p is the probability distribution function
- $0 \leq p(x) \leq 1$
- $\Sigma\, p(x) = 1$ (the sum is for all $x \in S$)
- $p(E') = 1 - p(E)$
- $p(E \cup F) = p(E) + p(F)$ for disjoint events

[2] Conditional Events:
The probability of E given F is $p(E \mid F)$.
- $p(E \mid F) = p(E \cap F) / p(F)$ conditional events
- $p(E \cap F) = p(E) * p(F)$ for independent events

[3] Probability of event E:

p(E) = the sum of the probabilities of the outcomes in E.

[4] Example:

A biased die (with 3 is twice loaded than other outcomes) is rolled. What is the probability of an odd number?

Solution:

p(E) = p(1) + p(3) + p(5) = 1/7 + 2/7 + 1/7 = 4/7.

[5] Uniform distribution:

In normal distribution of n outcomes the probability of each outcome is 1/n.
Examples: flipping a fair coin, and rolling a fair die.

[6] Bernoulli Trials:

Experiment with 2 outcomes: prob(success) = p, and prob(failure) = q, where q = 1- p

[7] Example:

A biased coin (with p(H) = 2/3) is flipped 7 times independently. What is the probability that exactly four heads come up?

Solution:

Number of possible outcomes = 128
Number of ways to get 4 H out of 7 coins = C(7,4)
Prob of each way has 4 H and 3 T = $(2/3)^4\ (1/3)^3$
Therefore, Prob(4 heads) = C(7, 4) $(2/3)^4\ (1/3)^3 = 35 * 16 / 3^7$

[8] Bernoulli Trials Formula:

The probability of exactly *k* successes in *n* independent Bernoulli trials is:

$$b(k;\ n,\ p) = C(n,\ k)\ p^k\ q^{n-k}$$

The function *b* is called the binomial distribution.

[9] Example:

A random bit string is generated, with p(0) = 90% and p(1) = 10%. What is the probability that a random string of ten bits contains exactly eight 0's?

Solution:

$$b(8;\ 10,\ 0.9) = C(10,\ 8)\ (0.9)^8\ (0.1)^2$$

[10] Example:

Using the binomial theorem, prove that the sum of the probabilities of the successes of n Bernoulli trials is 1. i.e. $\Sigma\ b(k;\ n,\ p) = 1$

Solution:

$$\sum_{k=0}^{n} C(n,\ k)\ p^k\ q^{n-k} = (p+q)^n \quad \text{by binomial theorem}$$

$$= (1)^n \quad \text{since } q = 1 - p$$

$$= 1$$