Cayley Theorem

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Recall: $(S_n, \circ)$

7] Thrm: $\forall \pi \in S_n$, π can be written as a product of disjoint cycles. The cycles that appear in the product are unique.

8] e.g. $\pi = [1\ 4\ 6\ 5\ 2\ 3] \in S_6$

		Cycle notation
cycles: $c_1 : 1 \to 1$	(unit cycle $\Longrightarrow |c_1| = 1$)	(1)
$c_2 : 2 \to 4 \to 5 \to 2$	$\Longrightarrow |c_2| = 3$	$(2\ 4\ 5)$
$c_3 : 3 \to 6 \to 3$	$\Longrightarrow |c_3| = 2$	$(3\ 6)$

$$\pi = c_1 \circ c_2 \circ c_3 = (1) \circ (2\ 4\ 5) \circ (3\ 6)$$

examples:

$$\pi(4) = c_1 \circ c_2 \circ c_3(4) = c_1(c_2(c_3(4))) = c_1(c_2(4)) = c_1(5) = 5$$

$$\pi(6) = c_1 \circ c_2 \circ c_3(6) = c_1(c_2(c_3(6))) = c_1(c_2(3)) = c_1(3) = 3$$

$$\pi(1) = c_1 \circ c_2 \circ c_3(1) = c_1(c_2(c_3(1))) = c_1(c_2(1))$$

$$= c_1(c_2(1))$$
$$= c_1(1)$$
$$= 1$$

9] Notation. for compactness:

① use juxtaposition: $c_1 c_2 = c_1 \circ c_2$

② omit the unit cycles:

e.g. $(1)\circ(2\ 4\ 5)\circ(3\ 6) = (2\ 4\ 5)(3\ 6)$

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③ by convention:

$$e = [1\ 2\ 3\ 4\ 5\ 6] = (1)(2)(3)(4)(5)(6) = (1)$$

10] Prop. if $\pi \in S_n$ is written as a product of disjoint cycles, then $\mathrm{ord}(\pi)$ is the lcm of the lengths of its cycles

11] e.g. Given $\pi = [1\ 4\ 6\ 5\ 2\ 3] = \underline{(2\ 4\ 5)}(3\ 6)$

Find $\mathrm{ord}(\pi)$

$$\pi^2 = [1\ 5\ 3\ 2\ 4\ 6] = (2\ 5\ 4)(3)(6)$$
$$\pi^3 = [1\ 2\ 6\ 4\ 5\ 3] = (2)(4)(5)(3\ 6)$$
$$\pi^4 = [1\ 4\ 3\ 5\ 2\ 6] = (2\ 4\ 5)(3)(6)$$
$$\pi^5 = [1\ 5\ 6\ 2\ 4\ 3] = (2\ 5\ 4)(3\ 6)$$
$$\pi^6 = [1\ 2\ 3\ 4\ 5\ 6] = (2)(4)(5)(3)(6) = (1)$$

$\therefore \mathrm{ord}(\pi) = 6$

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$$\pi = (2\ 5\ 4)(3\ 6)$$

$$\therefore \operatorname{ord}(\pi) = \operatorname{lcm}(3,2) = 6$$

12] e.g. Find a subgroup of S_7 of:

① order 3:

idea: take π of order 3 by prop [10]

Construct: $H = \{\pi, \pi^2, \pi^3 = e\}$

To choose π of order 3, take any cycle of length 3, $(2\ 3\ 1)$, then we have

$$\pi = [2\ 3\ 1\ 4\ 5\ 6\ 7]$$

$$\therefore H = \{[\underline{2\ 3\ 1}\ 4\ 5\ 6\ 7], [3\ 1\ 2\ 4\ 5\ 6\ 7], [1\ 2\ 3\ 4\ 5\ 6\ 7]\}$$

② order 10

Take two cycles of lengths 5 and 2

$$\pi = (2\ 3\ 4\ 5\ 1)(7\ 6) \Rightarrow \pi^{10} = e$$

$$\therefore H = \{\pi, \pi^2, \pi^3, \ldots, \pi^{10}\}$$

See e.g. [12] in LNGT §3

13] Thm: (Cayley Theorem)

Every group is isomorphic to a subgroup of some permutation group.

14] e.g. $(\mathbb{Z}_2, +) \cong S_2$, by $\theta: \mathbb{Z} \to S_2$

$$\theta(0) = [1\ 2], \quad \theta(1) = [2\ 1]$$

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$(\mathbb{Z}_3, +) \cong H_3 \subseteq S_3$, $\theta : \mathbb{Z}_3 \to H$

$H = \{[1\ 2\ 3], [2\ 3\ 1], [3\ 1\ 2]\}$

$\theta(0) = [1\ 2\ 3]$

$\theta(1) = [2\ 3\ 1]$

$\theta(2) = [3\ 1\ 2]$

Read LNGT §3

End of Group Theory