Closures of Relations

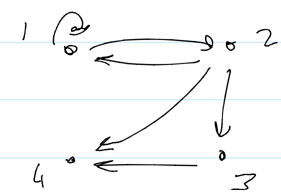
Wednesday, March 16, 2022 12:43 PM

Recall : relation properties

Missing:
#12, 22, 24, 26
34.

Objective: to answer these questions
① is R reflexive? what is missing?
② is R symmetric? what is missing?
③ is R transitive? What is missing?

1] e.g. 1. let $R = \{(1,2), (1,1), (2,1), (2,3), (2,4), (3,4)\}$ be a relation on $A = \{1,2,3,4\}$



① is R reflexive? what is missing?
No, missing $\Delta = \{(2,2), (3,3), (4,4)\}$

② is R symmetric? what is missing?
No, for $\Delta = \{(3,2), (4,2), (4,3)\}$

③ is R transitive? What is missing?
No, for $\Delta = \{(1,4), (1,3), (2,2), ..\}$

2] Def^n. the relation S is the closure of the relation R with respect to property P if:
① $R \subseteq S$
② S has property P, and
③ $\forall T$, if T satisfies ① and ② then $S \subseteq T$

i.e. S is the p-closure of R if $S = R \cup \Delta$ has property P with the smallest Δ, (gives the smallest S)

3] e.g. for R in e.g. 1.,
① the reflexive-closure of R is:

$S = R \cup \Delta = \{ (1,2), (1,1), (2,1),$
$(2,3), (2,4), (3,4),$
$(2,2), (3,3), (4,4) \}$

② the symmetric-closure of R is:

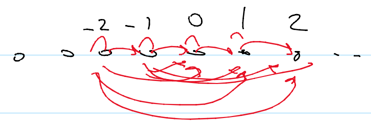
$S = R \cup \Delta = \{ (1,2), (1,1), (2,1),$
$(2,3), (2,4), (3,4),$
$(3,2), (4,2), (4,3) \}$

③ the transitive-closure of R is:

$S = R \cup \Delta = \{ (1,2), (1,1), (2,1),$
$(2,3), (2,4), (3,4),$
$(1,4), (1,3), (2,2) \}$

4] Exer: $R = \{ (a,b) \mid a < b \}$ on $\mathbb{Z}$

find the reflexive-closure of R.



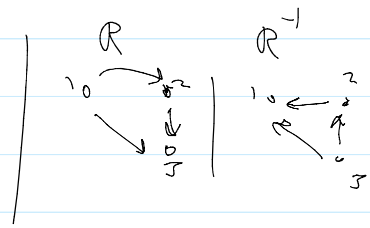
Sol.

$S = R \cup \Delta$, where $\Delta = \{ (a,a) \mid a \in \mathbb{Z} \}$

$\therefore S = \{ (a,b) \mid a \leq b \}$ on $\mathbb{Z}$.

5] Def^n. the inverse of relation R is

$$R^{-1} = \{ (b,a) \mid a R b \}$$



e.g. $R = \{ (a,b) \mid a < b \}$

then $R^{-1} = \{ (a,b) \mid a > b \}$

6] e.g. Find the symmetric-closure of $R = \{ (a,b) \mid a < b \}$

Sol. $S = R \cup \Delta$, where $\Delta = R^{-1}$

$\therefore S = \{ (a,b) \mid a \neq b \}$

7] Thrm: the symmetric-closure of R is $S = R \cup R^{-1}$

8] Def^n. let R be a relation on A,

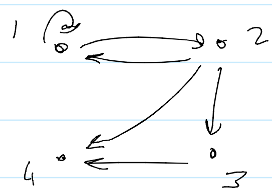
① R^n is the relation on A s.t. $x R^n y$ if there is a path from x to y of length n on R.

② R^* is the connectivity relation

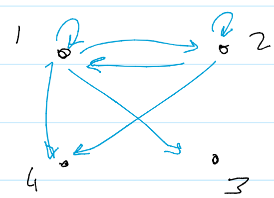
i.e. $x R^* y$ if there is a path in R from x to y.

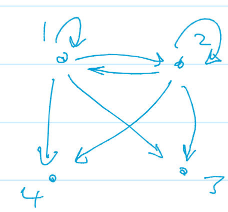
$$R^* = R \cup R^2 \cup R^3 \cup \cdots = \bigcup_{i=1}^{\infty} R^i$$

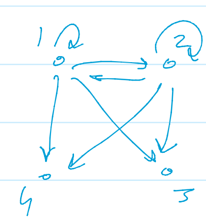
9] e.g.

R: 

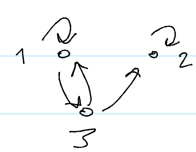
find R^2, R^3, R^4, ...

R^2 

R^3: 

R^4: 

10] Thrm: R^* is the transitive-closure of R

11] e.g. let $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ 

Find the transitive-closure of R

Sol.

$$M_{R^*} = M_R \vee M_R^2 \vee M_R^3 \vee \cdots$$

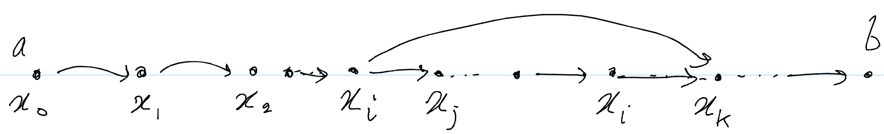
$M_R^2 = \begin{bmatrix} 1 & 1 & \\ 0 & 1 & \\ 1 & & \end{bmatrix}$ $M_R^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

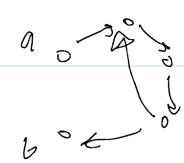
$\therefore M_R^* = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ ← the transitive-closure of R

03-28

12] To compute the transitive-closure :

Lemma. Let R be a relation on A, $|A| = n$,
If there is a path from a to b of length $> n$,
then there is path from a to b of length $\leq n$,
if $a \neq b$, the length $< n$.

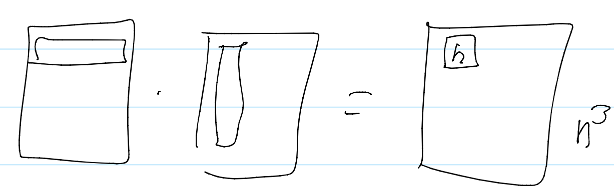




13] Alg 1 (Matrix Multiplication)

$A = M_R$
$B = A$
for $i = 2$ to n
  $A = A \odot M_R$
  $B = B \vee A$
Return B ← M_R^*

Time complexity



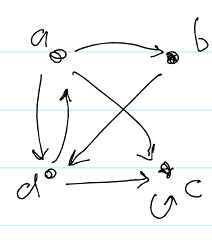
time $= O(n^4)$

14] Alg 2. (Warshall's Algorithm)

$W = M_R$
for $k = 1$ to n
  for $i = 1$ to n
    for $j = 1$ to n
      $W_{ij} = W_{ij} \vee (W_{ik} \wedge W_{kj})$

$M_R = W$

$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$



$k = 1 = a$

for $j = 1$ to n

$$W_{ij} = W_{ij} \vee (W_{ik} \wedge W_{kj})$$

Return W (or W_k, for $k = n$)

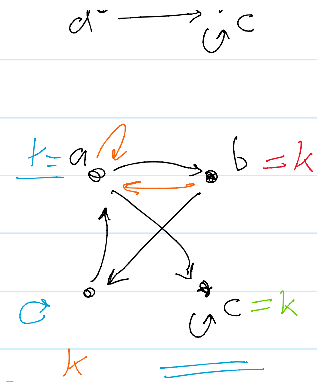
Time complexity:

$$O(n^3)$$

$[1\ 0\ 1\ 0]$

$k = 1 =$

$$W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 1$$

$W_3 = W_2$,

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \leftarrow \text{transitive closure}$$