Solving Linear Recurrence Relations

Monday, May 4, 2026 11:12 AM

Missing:
700# 2, 9, 12, 17, 22, 34
800# 1, 3, 10,
+B# 802, 718

Recall Recurrence Relations

§ 8.2 Solving Linear Recurrence Relations

1] Def^n. a Linear Homogeneous Recurrence Relation with Constant Coefficients (LHRRCC) of degree k is of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Here:

Linear: sum of multiples of previous terms (no power)

homogeneous: all terms contain a_j

constant coefficients: $c_1, c_2, \ldots, c_k \in \mathbb{R}$

degree k: a_n is computed in terms of previous k terms $(c_k \neq 0)$

e.g. ① Fibonacci: $a_n = a_{n-1} + a_{n-2}$, is LHRRCC of deg 2.

② $a_n = a_{n-5} + a_{n-3}$ is LHRRCC of deg 5

③ Hanoi: $H_n = 2H_{n-1} + 1$ is Linear, but not homogeneous.

2] Def^n. Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ be a LHRRCC.

The solution has the form $a_n = r^n$

$$\Rightarrow r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

div by r^{n-k}

$$\Rightarrow r^k = c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k$$

$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0 \leftarrow \text{characteristic equation}$$

$\therefore \{a_k\}$ with $a_n = r^n$ is a solution iff r is a root.

characteristic roots

3] Thrm 1:

Let $c_1, c_2 \in \mathbb{R}$,

$a_n = c_1 a_{n-1} + c_2 a_{n-2}$

if $r^2 - c_1 r - c_2 = 0$ has two roots, r_1 and r_2, then the solution is of the form:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \text{for} \quad n \geq 0, \ \alpha_1 \text{ and } \alpha_2 \in \mathbb{R}$$

4] e.g. Find the sol. of the recurrence relation:

$$a_n = a_{n-1} + 2a_{n-2} \ ; \qquad a_0 = 2, \ a_1 = 7$$

Sol^n.

5] (How to solve?)

① Find the char. eq^n

$$r^2 - r - 2 = 0$$

② Find the char. roots

$$(r-2)(r+1) = 0$$

$$\Rightarrow r_1 = 2, \quad r_2 = -1$$

③ Apply the theorem:

Thrm 1: $\quad a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

④ Use the initial conditions to solve for α_1 and α_2

$$\left.\begin{array}{l} a_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 \Rightarrow 2 = \alpha_1 + \alpha_2 \\ a_1 = \alpha_1 r_1^1 + \alpha_2 r_2^1 \Rightarrow 7 = \alpha_1(2) + \alpha_2(-1) \end{array}\right] \xrightarrow{\text{Solve}} \begin{array}{l} \alpha_1 = 9/3 = 3 \\ \alpha_2 = -1 \end{array}$$

⑤ The solution is $\quad a_n = 3(2)^n - (-1)^n$

Verify: $\quad a_2 = 7 + 2(2) = 11$

$a_n = 3(2^2) - (-1)^2 = 12 - 1 = 11$ ✓

6] Thrm 2 : Let $c_1, c_2 \in \mathbb{R}$,

if $r^2 - c_1 r - c_2 = 0$ has only one root r_0 then the solution is :

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

where α_1, α_2 are constants

7] Exer Find the solution of

$$a_n = 6 a_{n-1} - 9 a_{n-2} \; ; \quad \text{with } a_0 = 1, \; a_1 = 6$$

Solⁿ. ① Find char. eqⁿ : $r^2 - 6r + 9 = 0$

② Find the roots: $(r-3)^2 = 0$

$\Rightarrow r_0 = 3$

③ Apply Thrm 2 :

$$a_n = \alpha_1 r_0^n + \underline{\alpha_2 n r_0^n} \quad \text{for } n \geq 0$$

④ Solve for α_1, α_2 :

$a_0 = 1 = \alpha_1 r_0^0 + \alpha_2 (0) r_0^0 \Rightarrow \alpha_1 = 1$

$a_1 = 6 = \alpha_1 r_0^1 + \alpha_2 (1) r_0^1 \Rightarrow 6 = 3 + \alpha_2 \cdot 3 \Rightarrow \alpha_2 = 1$

⑤ The solⁿ is : $a_n = 3^n + n(3^n)$

Verify : $a_2 = 6(6) - 9(1) = 36 - 9 = 27$

$a_2 = 3^2 + 2(3^2) = 9 + 18 = 27$ ✓ +B#802