Recurrence Relations

Wednesday, April 29, 2026 11:14 AM

Recall: Counting

1) Exer.

F13| How many positive integers ≤ 100 are

① divisible by 3?

$$\left\lfloor \frac{100}{3} \right\rfloor = 33$$

② divisible by 7?

$$\left\lfloor \frac{100}{7} \right\rfloor = 14$$

③ divisible by 3 and 7?

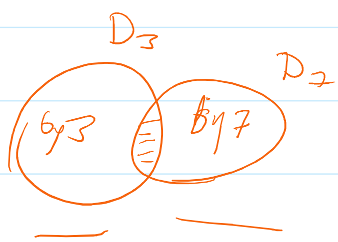
$$\left\lfloor \frac{100}{21} \right\rfloor = 4$$

④ divisible by 3 or 7?

$$|D_3 \cup D_7| = |D_3| + |D_7| - |D_3 \cap D_7|$$

$$33 + 14 - 4 = 43$$

+B# 727



Mission

700# 4, 8, 12, 15, 25, 28 Ex

Here: #808

800# 1, 14

Here #708

+B# 727, 702, 703

F14| ⑤ divisible by 5 or 7 and between 200 and 400. $(200 \leq n \leq 400)$

$$\boxed{1 \text{ --- } 400} - \boxed{1 \text{ --- } 199}$$

$$\left(\left\lfloor \frac{400}{5} \right\rfloor + \left\lfloor \frac{400}{7} \right\rfloor - \left\lfloor \frac{400}{35} \right\rfloor\right) - \left(\left\lfloor \frac{199}{5} \right\rfloor + \left\lfloor \frac{199}{7} \right\rfloor - \left\lfloor \frac{199}{35} \right\rfloor\right)$$

$$80 + 57 - 11 \qquad 39 + 28 - 5$$

$$126 - 62 = 64$$

62
A 63 ✓
W 64 -
R 65
66

§ 8.1 Recurrence Relations

2] Def^n. a recurrence relation expands a_n in terms of a_i, for $i < n$

e.g. $a_n = 2a_{n-1}$; $\underbrace{a_0 = 1}_{\text{initial condition}}$

$\Rightarrow \{a_n\} = 1, 2, 4, 8, \ldots$

3] Def^n Given a sequence $\{a_n\}$, a recurrence relation is an equation of the form $a_n = f(a_1, a_2, \ldots, a_{n-1})$, for $n \geq 0$. A solusion of this $\{a_n\}$ is a sequence that satisfies this equation

e.g.

Fibonacci Number :

$a_n = a_{n-1} + a_{n-2}$;

$a_0 = 1$, $a_1 = 1$

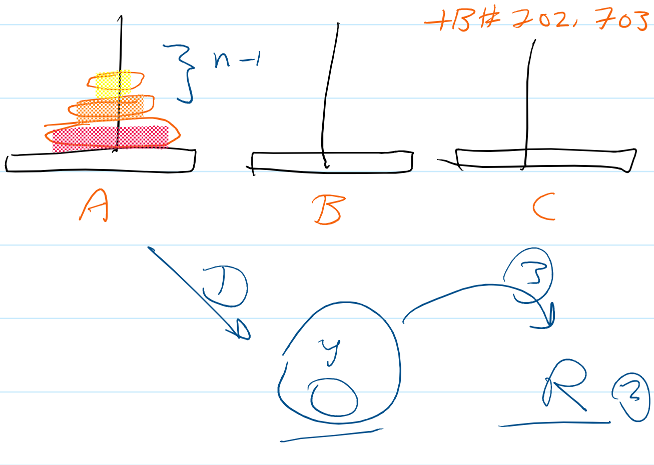
$\{a_n\} = 1, 1, 2, 3, 5, 8, 13, \ldots$

4] Tower of Hanoi

+B# 202, 703

$H_n = H_{n-1} + 1 + H_{n-1}$

$= 2H_{n-1} + 1$; $H_1 = 1$



5] e.g. Solve H_n by expansion:

$$H_n = 2H_{n-1} + 1$$
$$= 2(2H_{n-2}+1)+1 \quad = 2^2 H_{n-2} + 2 + 1$$
$$= 2^2(2H_{n-3}+1) + 2 + 1$$
$$= 2^3 H_{n-3} + 2^2 + 2^1 + 2^0$$
$$\vdots$$
$$= 2^i H_{n-i} + 2^{i-1} + 2^{i-2} + \dots + 2^0$$

for $i = n-1$

$$= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \dots 2^0$$
$$= \sum_{i=0}^{n-1} 2^i = 2 \cdot 2^{n-1} - 1 = 2^n - 1$$

6] e.g. How many bit strings of length n are there without consecutive 1's

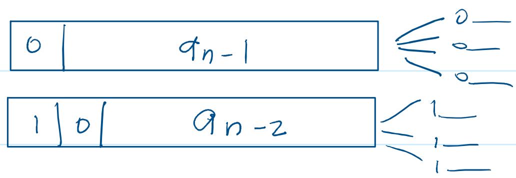
Sol^n. let $\underline{a_n}$ be the number of bit strings of length n without consecutive 1's.

$a_1 = 2$ $\boxed{\begin{matrix}0\\1\end{matrix}}$

$a_2 = 3$ $\boxed{\begin{matrix}00\\01\\10\end{matrix}}$

$a_n = a_{n-1} + a_{n-2}$; $n \geq 3$

a_n :



$\{a_n\}$: 2, 3, 5, 8, 13, 21, 34, 55, ...

7] Exer :

How many n-digit numbers are there without "99" as a substring?

095229 ✓

0991123 ×

$a_1 = 10$

$a_2 = 99$

a_n

$\boxed{9 \mid x \mid a_{n-2}}$ 9, 9, 9

$\boxed{x \mid a_{n-1}}$ 1, 8, 7

$$a_n = 9 \cdot a_{n-1} + 9 \cdot a_{n-2}$$

$\{a_n\}$: 10, 99, $9(99+10) = 972$, ...