Binomial Theorem

Monday, April 27, 2026 11:05 AM

Recall : Permutations & Combinations

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

$$\binom{10}{10} = \frac{\cancel{10!}}{(10-10)!\,\cancel{10!}} = \frac{1}{1} = 1$$

$P(5,5) = 5! = 120$; $P(5,4) = 5.4.3.2 = 120$

Missing
700 # 4, 28
Here # 811 , 801
800 # 3, 6, 8, 20
+B # 722
+B# 810, 822

1] Exer : In how ways can 5 SWE students
① 3 COE student line up for a photo ?

$P(n,r)$
$C(n,r)$
$n!$, n^r

Sol^n. $8!$

F B D R H 2 3 k

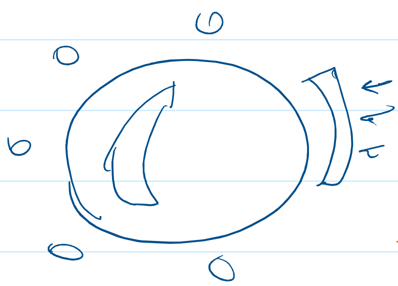
② COE students are not adjacent ?

H R D R B

Sol^n. for SWE : $5!$
for COE : $P(6,3) = \binom{6}{3}.3!$
Total : $5! \cdot P(6,3)$ by product rule

③ Sit on a round table , all COE students are adjacent.

$5! . 3!$



$4! \times 5.3!$
$= 5! . 3!$
+B# 810, 822

§ 6.4 Binomial Theorem

2] e.g.

$$(x+y)^3 = (x+y)(x+y)(x+y)$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$
$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$\begin{array}{l} (x+y) \\ \underline{(x+y)} \\ xx + xy + yx + yy \\ \underline{(x+y)} \\ xxx + xxy + xyx + xyy \\ yxx + yxy + yyx + yyy \end{array}$$

3] Thrm: (Binomial Theorem)

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

4] e.g. $(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

$$= x^4 + 4x^3y + \underbrace{6x^2y^2}_{\text{term}} + \underset{\text{coefficent}}{4}xy^3 + y^4$$

5] e.g. What is the coefficient of xy^7 in $(x+y)^8$

$$(x+y)^8 = \cdots\cdots + \underbrace{\binom{8}{7}xy^7}_{\text{term}} + \cdots$$

$\therefore$ the coeff. is $\binom{8}{7} = 8$

6] exer: Find the coeff. of $x^{12}y^{13}$ in $(2x-3y)^{25}$

Sol^n. $\quad \cdots + \binom{25}{13}(2x)^{12}(-3y)^{13} + \cdots$

$$\binom{25}{13} 2^{12} (-3)^{13} x^{12} y^{13}$$

$\therefore$ coeffi is: $-3 \cdot 6^{12} \binom{25}{13}$

7] Corollary 1 : $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Proof: $(1+1)^n = 2^n = RHS$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} = LHS$$

e.g.

Prove that if $|A| = n$, then $|P(A)| = 2^n$

$$|P(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

$$= \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

8] Col. 2 :

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof $(1-1)^n = RHS$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = LHS$$

9] Col. 3 :

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Proof: $(1+2)^n = 3^n = RHS$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} 2^k \binom{n}{k} = LHS$$

e.g. $\binom{3}{0} + 2\binom{3}{1} + 2^2\binom{3}{2} + 2^3\binom{3}{3} = 3^n$

10] Pascal Triangle and identity

Pascal Identity:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \dots$$

11] Tennis Problem

100 players

- Bijection method

Observation: one game sends one player out

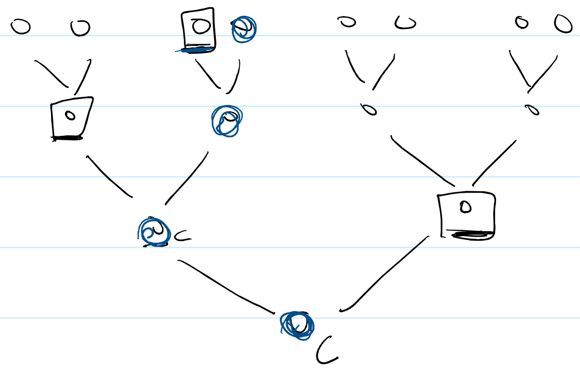
To send 99 players out, we need 99 games

To Find 1st and 2nd placess

$99 + 6 = 105$

TB 722

For 8 players



8

$7 + 2 = 9$

12] Exer: K & K counting problems

6 people

① How many solutions are there?

Soln 2^6

② if one says: at least one of us is a knave

Soln. $2^5 - 1$

$T \underbrace{\_\ \_\ \_\ \_\ \_}_{\text{all } 2^5 \text{ are possible except TTTTT}}$

$\therefore 2^5 - 1$