ICS 253

Presents

Mathematical Induction

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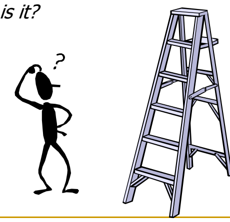
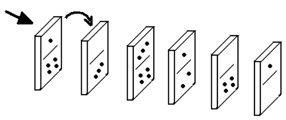
Mathematical Induction

What is it?



Mathematical Induction



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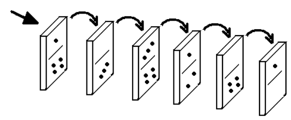
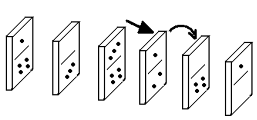
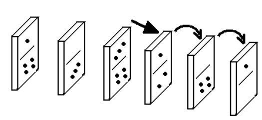


Mathematical Induction



Mathematical Induction



Dominos Effect

Mathematical Induction

The First Principle of Induction:

Let P(n) be an open sentence about positive integers, 1,2,3,… and assume the following:

(a) P(1) is a true statement.

(b) $\forall$ k ≥ 1, if P(k) is true then P(k+1) is true.

Then, we conclude that P(n) is true for all n ≥ 1

Mathematical Induction

So what?

We can prove statement P(n) is true for all n in simple two steps:

1. Basis step: P(1)
2. Inductive step: P(k) → P(k+1)

Mathematical Induction

Example:

What is the sum of the first n positive integers?

SUM(n) = 1+2+3+…+n = ?

What is SUM(3)?

SUM (3) = 1 + 2 + 3 = 6

Mathematical Induction

Problem:

Find SUM(n) = ??

Mathematical Induction

Solution:

SUM(n) = n(n+1) / 2

Prove it !!

Well, we know SUM(3) = 1+2+3 = 6

Try RHS, 3(3+1) / 2 = 3*4 / 2 = 6

OK ??

But this is not a proof for all n.

Mathematical Induction

Problem:

Prove the statement P(n) for all n.

P(n): SUM(n) = n(n+1) / 2

Example, P(5) read:

The sum of the first five integers is 5(5+1)/2

Mathematical Induction

OK, try this:

P(1): SUM(1) = 1, and 1(2)/2 = 1.

P(2): 1+2 = 3, and 2(3)/2 = 3.

P(3): 1+2+3=6, and 3(4)/2=6.

P(4): 1+2+3+4=10, and 4(5)/2=10.

P(5): 1+2+3+4+5=15, and 5(6)/2=15.

and so on… Therefore, P(n) is true for all n.

Do you buy this proof??!

Mathematical Induction

Prove SUM(n) = n(n+1)/2

Observation:

SUM(k) = 1+2+3+…+ k

SUM(k+1) = 1+2+3+…+ k + (k+1)

So, SUM(k+1) = SUM(k) + (k+1)

There is a relation between P(k) and P(k+1)

P(k): SUM(k) = k(k+1)/2

P(k+1): SUM(k+1) = (k+1)(k+2)/2

Show that P(k) → P(k+1)

Prove SUM(n) = n(n+1)/2

SUM(k+1) = SUM(k) + (k+1)

P(k): SUM(k) = k(k+1)/2

P(k+1): SUM(k+1) = (k+1)(k+2)/2

Show that P(k) → P(k+1)

Prove SUM(n) = n(n+1)/2

SUM(k+1) = SUM(k) + (k+1)

P(k): SUM(k) = k(k+1)/2

P(k+1): SUM(k+1) = (k+1)(k+2)/2

Show that P(k) → P(k+1)

SUM(k+1) = SUM(k) + (k+1)

= k(k+1)/2 + (k+1)

= k(k+1)/2 + 2(k+1)/2

= [k(k+1) + 2(k+1)] /2

= [k(k+1) + 2(k+1)] /2

= [(k+1) (k+2)] /2

Therefore, if SUM(k) = k(k+1)/2 then SUM(k+1) = (k+1)(k+2)/2

So, P(k) → P(k+1) ←(This is called the inductive step)

We know that P(1) is true. What else can we imply?

Mathematical Induction

- To prove statement P(n) for n = 1,2,3,...
- Basis step: Show P(1) is true.
- Inductive Step:
  - Assume P(k) is true.
  - Show that P(k+1) is true based on the assumption.
- This is called the **first principle of induction**.
- In the **second principle of induction** (or **strong induction**), we assume that P(j) is true for all j ≤ k and then we prove P(k+1).

Exam Tip: Prove by induction (First Principle)

- Show that 1+2+ … + n = n(n+1)/2 using induction.
- Solution Template (First Principle ONLY)

P(n): 1+2+ … + n = n(n+1)/2

Basis Step: P(1) is true because … (~25%)

Inductive Step:

Assume P(k) is true, 1+2+…+k = k(k+1)/2

Now, we show that P(k+1) is true. (~25%)

In this part you must use the assumption.

By induction principle, P(n) is true for all n ≥ 1.

Example:

- Prove $11^n – 6$ is divisible by 5, for n=1,2,3…

P(n): $11^n – 6$ is divisible by 5
Basis Step: for n = 1,
P(1): $11^1 – 6 = 5$ which is divisible by 5.
So, P(1) is true.
Inductive Step:
Assume P(k) is true, $11^k – 6$ is divisible by 5.
We show that P(k+1) is also true.
i.e. $11^{k+1} – 6$ is also divisible by 5
---------- ~50% is obtained upto this line ---------
(Remember: use the assumption here)

Example:
P(n): $11^n – 6$ is divisible by 5

Inductive Step:
Assume P(k) is true, $11^k – 6$ is divisible by 5.
We show that S(k+1):$11^{k+1} – 6$ is divisible by 5

$$
\begin{aligned}
11^{k+1} - 6 &= 11 . 11^k - 6 \\
&= (10+1) . 11^k - 6 \\
&= 10 . 11^k + 11^k - 6 \\
&= 10 . 11^k + 11^k - 6
\end{aligned}
$$

But $10 . 11^k$ is divisible by 5,
and $11^k – 6$ is divisible by 5 (by assumption).
Therefore, $11^{k+1} – 6$ is divisible by 5,
and hence P(k+1) is true.
By mathematical induction, P(n) is true for all $n \geq 1$

Exam Tip:
Prove by induction (Second Principle)

- E.g. Show that the statement is true for all n=1,2,3,…
- Solution Template
- P(n): put the statement

Basis Step:
Show that P(1), P(2),.. are true by testing/substitution … (~25%)
Inductive Step:
Assume P(j) is true for all $j \leq k$
Now, we show that P(k+1) is true. (~25%)

In this part you must use the assumption.

By induction principle P(n) is true for all $n \geq 1$.

Example on Second Principle.

- Prove that any integer n > 1 can be written as the product of primes
- P(n): n can be written as the product of primes.
- Basis Step:
  - P(2) is true since 2 is prime
- Inductive Step:
  - Assume P(j) is true for all $j \leq k$

Example on Second Principle.

- Want to show: P(k+1)
- Case 1:
  if k+1 is prime, then P(k+1) is true and we are done.
- Case 2:
  if k+1 is composite, then there are two integers, a, b, such that k+1 = a.b and $2 \leq a \leq b < k+1$
- By assumption, P(a) and P(b) are true, which implies that a and b can be written as the products of prime.
  Therefore, a.b = k+1 can be written as the product of primes.
- By induction principle, all $n \geq 2$ can be written as the product of primes.

Mathematical Induction

First step

Inductive Step