

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 2.4—Sequences and Summations



— Page references correspond to locations of Extra Examples icons in the textbook.

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#1. Find a rule that produces a sequence a_1, a_2, a_3, \dots with the first terms 5, 7, 9, 11, 13, \dots

Solution:

This is the sequence of odd positive integers, beginning with 5. Each odd positive integer has the form $2n + 1$. Because we need $a_1 = 5$, we add 3, not 1. Therefore $a_n = 2n + 3$.

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#2. Find a formula for an infinite sequence a_1, a_2, a_3, \dots that begins with the terms $1/3, 1/4, 1/5, 1/6, \dots$

Solution:

The sequence behaves like the sequence whose terms are $1/n$, except that we begin with $a_1 = 1/3$ rather than $a_1 = 1/1$. Therefore, $a_n = 1/(n + 2)$.

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#3. Find a formula for an infinite sequence a_1, a_2, a_3, \dots that begins with the terms 7, 11, 15, 19, 23, \dots

Solution:

Each term is a multiple of 4, with 1 subtracted. The first term is “4 times 2 minus 1”, the second term is “4 times 3 minus 1”, etc.. Therefore the n th term is $a_n = 4(n + 1) - 1 = 4n + 3$.

The first few terms can be checked: $a_1 = 4 \cdot 1 + 3 = 7$, $a_2 = 4 \cdot 2 + 3 = 11$, $a_3 = 4 \cdot 3 + 3 = 15$, etc.

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#4. Find a formula for an infinite sequence a_1, a_2, a_3, \dots that begins with the terms 1, 2, 1, 2, 1, 2, 1 and continues this alternating pattern.

Solution:

The terms alternate between 1 and 2. We can look on this as beginning with a “central number” 1.5 and alternately subtracting 0.5 from 1.5 and adding 0.5 to 1.5. A method of alternately adding and subtracting the same number involves using powers of -1 . We can alternately subtract 0.5 from 1.5 and add 0.5 to 1.5 by using $1.5 + 0.5(-1)^n$. Thus, $a_n = 1.5 + 0.5(-1)^n$.

Note: This is not the only formula for the given sequence. For example, we could use $a_n = ((n+1) \bmod 2) + 1$.

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#5. Find a formula for an infinite sequence a_1, a_2, a_3, \dots that begins with the terms 0, 2, 6, 12, 20, 30, 42, \dots

Solution:

This sequence increases at an increasing rate, which suggests n^2 as a possibility. If we write the first terms of the n^2 sequence, we have 1, 4, 9, 16, 25, 36, 49, \dots . The terms of this sequence of squares differ from the terms of the given sequence by 1, 2, 3, 4, 5, \dots . This gives a formula for the given sequence: $a_n = n^2 - n$.

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#6. Find a rule that produces a sequence a_1, a_2, a_3, \dots with the first terms 3, 6, 12, 24, 48, \dots

Solution:

After the first term, each term is double the previous term. This suggests that a formula is $3(2^n)$. However, this does not work because this rule gives $a_1 = 3(2^1) = 6$. In order to have $a_1 = 3$, we need to reduce the exponent by 1: $a_n = 3(2^{n-1})$.

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#7. Find a rule that produces a sequence a_1, a_2, a_3, \dots with the first terms 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \dots

Solution:

This sequence grows, but at half the rate of $a_n = n$. If we try $a_n = n/2$, we obtain the sequence $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \dots$. Round up each of these terms to get 1, 1, 2, 2, 3, 3, 4, 4, \dots . Therefore a formula for the sequence is $a_n = \lceil n/2 \rceil$.

p.154, icon at Example 9

#1. Express in sigma notation the sum of the first 50 terms of the series $4 + 4 + 4 + 4 + 4 + \dots$

Solution:

In sigma notation we have $\sum_{i=1}^{50} 4$. This series tells us to add fifty 4's — one 4 when i is 1, one 4 when i is 2, one 4 when i is 3, etc. Note: It is not correct to write $\sum_{i=1}^{50} 4i$, which would be $4 + 8 + 12 + \dots + 200$.

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#2. Find the value of each of these sums

$$(a) \sum_{j=1}^4 (j^2 - 1).$$

$$(b) \sum_{k=1}^4 (k^2 - 1).$$

$$(c) \sum_{j=1}^4 (k^2 - 1).$$

Solution:

$$(a) \sum_{j=1}^4 (j^2 - 1) = (1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) = 26.$$

(b) The variable used in the summation process does not matter, so the sum is identical to that in part (a):

$$\sum_{k=1}^4 (k^2 - 1) = (1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) = 26,$$

(c) In this case the variable of summation, j , does not appear in the definition of the terms. The letter k is a constant. When $j = 1$, the term is $k^2 - 1$; when $j = 2$, the term is $k^2 - 1$; when $j = 3$, the term is $k^2 - 1$; and when $j = 4$, the term is $k^2 - 1$. Therefore

$$\sum_{j=1}^4 (k^2 - 1) = (k^2 - 1) + (k^2 - 1) + (k^2 - 1) + (k^2 - 1) = 4k^2 - 4.$$

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#3. Find the value of each of these sums:

$$(a) \sum_{k=1}^4 (k^2 - 1).$$

$$(b) \sum_{k=1}^4 k^2 - 1.$$

Solution:

$$(a) \sum_{k=1}^4 (k^2 - 1) = (1^2 - 1) + (2^2 - 1) + (3^2 - 1) + (4^2 - 1) = 0 + 3 + 8 + 15 = 26.$$

(b) Note that only the terms k^2 are summed. After this sum is found, then 1 is subtracted.

$$\sum_{k=1}^4 k^2 - 1 = 1^2 + 2^2 + 3^2 + 4^2 - 1 = 29. \text{ (It matters whether or not parentheses are placed around the terms in the expressions being added.)}$$

p.155, icon at Example 12

#1. Express in sigma notation the sum of the first 50 terms of the series $3 + 6 + 9 + 12 + 15 + \dots$

Solution:

In sigma notation we have $\sum_{i=1}^{50} 3i$. Note that we could also write this in other forms, for example $\sum_{j=1}^{50} 3j$ or $\sum_{k=1}^{50} 3k$ (we can use any variable as the index of summation). We can also change the limits of summation, obtaining forms such as the sum $\sum_{i=0}^{49} 3(i+1)$. Note: It is not correct to write $\sum_{i=1}^{50} (3+i)$; this represents the sum $4 + 5 + 6 + \dots + 53$.

p.155, icon at Example 12

#2. The following is a geometric series: $\sum_{i=0}^{10} 2^i$. Identify a , r , and n , and then find the sum of the series.

Solution:

Written out in expanded form, the series is $2^0 + 2^1 + 2^2 + \dots + 2^{10}$. Therefore $a = 2^0 = 1$, $r = 2$, and $n = 10$. Using the formula for the sum, we have $\sum_{i=0}^{10} 2^i = \frac{a(r^{n+1} - 1)}{r - 1} = 2^{11} - 1 = 2,047$.

p.155, icon at Example 12

#3. The following is a geometric series: $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{64}$. Identify a , r , and n , and then find the sum of the series.

Solution:

$a = 4$ and $r = 1/2$. To find n it helps to rewrite the series as $4 + 4 \cdot \frac{1}{2} + 4 \cdot (\frac{1}{2})^2 + 4 \cdot (\frac{1}{2})^3 + \dots + 4 \cdot (\frac{1}{2})^8$. Therefore $n = 8$. (It is a common mistake to take the last term, $\frac{1}{64}$, and write it as $\frac{1}{2^6}$ and conclude that $n = 6$. To use the formula for the sum of a geometric series, we need to write the last term as ar^n , not simply r^n .)

Using the formula for the sum of a geometric series, we obtain the sum

$$\frac{a(r^{n+1} - 1)}{r - 1} = \frac{4((\frac{1}{2})^{8+1} - 1)}{\frac{1}{2} - 1} = \frac{4(-\frac{511}{512})}{-\frac{1}{2}} = 4 \cdot \frac{511}{256} = \frac{511}{64}.$$

p.155, icon at Example 12

#4. Find the sum of the series $2^4 + 2^5 + 2^6 + \dots + 2^{17}$.

Solution:

This is a geometric series with $a = 2^4$, $r = 2$, and $n = 13$. Therefore the sum is $\frac{2^4(2^{14} - 1)}{2 - 1} = 262,128$.

Alternately, we can write $2^4 + 2^5 + 2^6 + \cdots + 2^{17} = 2^4(1 + 2 + 2^2 + \cdots + 2^{13}) = 2^4 \cdot \frac{2^{14} - 1}{2 - 1} = 2^{18} - 2^4 = 262,128$.

p.157, icon at Example 16

#1. Find $1 + x^2 + x^4 + x^6 + x^8 + \cdots$ assuming $|x| < 1$.

Solution:

This is an infinite geometric series with $a = 1$ and $r = x^2$. Therefore the sum is $\frac{a}{1 - r} = \frac{1}{1 - x^2}$ and we have $1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1 - x^2}$.

p.157, icon at Example 16

#2. Prove that $\sum_{i=1}^{\infty} \frac{1}{4^i} = 2 \sum_{i=1}^{\infty} \frac{1}{7^i}$.

Solution:

Both sums are geometric series. $\sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$ and $\sum_{i=1}^{\infty} \frac{1}{7^i} = \frac{\frac{1}{7}}{1 - \frac{1}{7}} = \frac{1}{6}$. Therefore, the sum on the left is equal to twice the sum on the right.

p.157, icon at Example 16

#3. Find the sum of each of these infinite series:

(a) $\sum_{i=1}^{\infty} \frac{1}{2^i}$.

(b) $\sum_{i=1}^{\infty} (-1)^i \frac{1}{2^i}$.

Solution:

(a) $a = 1/2$ and $r = 1/2$. Therefore the sum is $\frac{1/2}{1 - 1/2} = 1$.

(b) $a = -1/2$ and $r = -1/2$. Therefore the sum is $\frac{-1/2}{1 - (-1/2)} = -1/3$.

p.160, icon at Example 21

#1. We know that the set of rational numbers is countable. Are the irrational numbers (the real numbers that cannot be written as fractions a/b where a and b are integers and $b \neq 0$) also countable, or are they uncountable?

Solution:

We will give a proof by contradiction that the irrational numbers are uncountable.

Suppose the irrational numbers were countable; then they can be listed as b_1, b_2, b_3, \dots . But we know that the rational numbers are also countable, and hence can be listed as a_1, a_2, a_3, \dots . “Interlace” the two lists as $a_1, b_1, a_2, b_2, a_3, \dots$ to obtain a countable set. But this is equal to the set of real numbers, because every real number is either rational or irrational. This says that the set of real numbers is a countable set, which contradicts the fact that the real numbers form an uncountable set. Therefore the irrational numbers are uncountable.

p.160, icon at Example 21

#2. Show that the set $\{x \mid 0 < x < 1\}$ is uncountable by showing that there is a one-to-one correspondence between this set and the set of all real numbers.

Solution:

We first show that there is a one-to-one correspondence between the interval $\{x \mid -\pi/2 < x < \pi/2\}$ and \mathbf{R} . We can use the function $f(x) = \arctan x$ (the inverse tangent function), which is a one-to-one function from $\{x \mid -\pi/2 < x < \pi/2\}$ onto \mathbf{R} .

We can then use the function $g: (0, 1) \rightarrow (-\pi/2, \pi/2)$ defined by $g(x) = \frac{\pi}{2}(2x - 1)$ (which is a one-to-one correspondence) and form the composition $f \circ g: (0, 1) \rightarrow \mathbf{R}$.

This gives the desired one-to-one correspondence from the interval $(0, 1)$ to \mathbf{R} . Because we know that \mathbf{R} is uncountable, so is $(0, 1)$.
